Polar Coordinates 2 - MS

Q1.

1	Uses identity.	$2\sin\theta\cos\left(\theta-\frac{1}{4}\pi\right)=\sin\left(2\theta-\frac{1}{4}\pi\right)+\sin\left(\frac{1}{4}\pi\right)$	M1		
	Uses sine-cosine link.	$=\cos\left(\frac{1}{2}\pi-2\theta+\frac{1}{4}\pi\right)+\frac{1}{\sqrt{2}}$	M1		
	Obtains result.	$=\cos\left(2\theta-\frac{3}{4}\pi\right)+\frac{1}{\sqrt{2}}$ (AG)	A1	3	
	Sketches graph.	Closed loop through origin, in correct position.	B1		
	Obtains line of symmetry.	For line of symmetry $2\theta-\frac{3}{4}\pi=0\Rightarrow\theta=\frac{3}{8}\pi$.	B1B1	3	
	Uses area of sector formula.	$A=\frac{1}{2}\int_0^{\frac{3}{4}\pi}\left\{\cos^2\left(2\theta-\frac{3}{4}\pi\right)+\sqrt{2}\cos\left(2\theta-\frac{3}{4}\pi\right)+\frac{1}{2}\right\}\,d\theta$	M1		
	Rearranges.	$=\frac{1}{2}\int_0^{\frac{3}{4}\pi}\left\{\frac{1}{2}\cos\left[4\theta-\frac{3}{2}\pi\right]+\sqrt{2}\cos\left[2\theta-\frac{3}{4}\pi\right]+1\right\}\,d\theta$	A1		
	Integrates correctly.	$=\left[\frac{1}{16}\sin\left(4\theta-\frac{3}{2}\pi\right)+\frac{1}{2\sqrt{2}}\sin\left(2\theta-\frac{3}{4}\pi\right)+\frac{\theta}{2}\right]_0^{\frac{3}{4}\pi}$	dM1A1		
	Substitutes limits.	$=\left[-\frac{1}{16}+\frac{1}{4}+\frac{3}{8}\pi\right]-\left[\frac{1}{16}-\frac{1}{4}\right]$	dM1		
	Obtains given answer.	$=\frac{3}{8}(\pi+1)$ (AG)	A1	6	[12]
		N.B Method marks are dependent in final part. If $\frac{1}{2}$ factor missing throughout – award M's (Max 3) If $2\times\frac{1}{2}\int_0^{\frac{3}{8}\pi}r^2\,d\theta$, penultimate line is $=\left[\frac{3}{8}\pi\right]-\left[\frac{1}{8}-\frac{1}{2}\right]$			

Q2.

(i)	Uses area formula.	$\text{Area}=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}4e^{2\theta}\,d\theta=\left[e^{2\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$	M1		
	Obtains result.	$=e^{\pi}-e^{\frac{\pi}{3}}\quad(=20.3)$	A1	2	
(ii)	Uses arc length formula.	$\text{Arc length}=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{4e^{2\theta}+4e^{2\theta}}\,d\theta=2\sqrt{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}e^{\theta}\,d\theta$	M1A1		
	Obtains result.	$=2\sqrt{2}\left[e^{\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}=2\sqrt{2}\left[e^{\frac{\pi}{2}}-e^{\frac{\pi}{6}}\right]\quad(=8.83)$	A1	3	[5]

Q3.

	Differentiates wrt θ	$\frac{dr}{d\theta} = 2\cos\theta - 2\cos 2\theta$	M1		
	Equates to zero	$2c - 2(2c^2 - 1) = 0 \Rightarrow 2c^2 - c - 1 = 0$	A1		
	And solves equation	$\Rightarrow (2c+1)(c-1) = 0$	M1		
		$\Rightarrow c = -\frac{1}{2}$ or 1	A1		
	States required points on C	$\left(\frac{3}{2}\sqrt{3}, \frac{2}{3}\pi\right)$	A1	**(5)**	
	Sketches C.	Approximate shape and location Accurate scaling.	B1 B1	**(2)**	
	Uses $\frac{1}{2}\int r^2 d\theta$	Area $= \frac{1}{2}\int_0^{\frac{\pi}{4}} 4\sin^2\theta(1 - 2\cos\theta + \cos^2\theta)d\theta$	M1 M1		
	Obtains an integrable form	$= \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta - 4\cos\theta\sin^2\theta + \frac{1}{4}[1 - \cos 4\theta]\right) d\theta$	A1A1		
	Integrates	$= \left[\frac{5\theta}{4} - \frac{\sin 2\theta}{2} - 4\frac{\sin^3\theta}{3} - \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{4}}$	M1		
	Obtains result	$= \frac{5}{16}\pi - \frac{1}{2} - \frac{\sqrt{2}}{3}$ or 0.0103	A1	**(6)**	**[13]**

Q4.

	Circle sketched Cardioid – correct location and orientation – correct indentation near pole.	B1 B1B1 (3)
	$\left(a, \frac{\pi}{2}\right)$ and $\left(a, \frac{3\pi}{2}\right)$ (B1 for reverse, or $a = a(1 - \cos\theta) \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ seen.)	B1B1 (2)
	Area $= \frac{1}{2}\pi a^2 + 2 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2(1 - \cos\theta)^2 d\theta$ (Half circle + Area of sector)	B1M1
	$= \frac{1}{2}\pi a^2 + a^2\int_0^{\frac{\pi}{2}}(1 - 2\cos\theta + \cos^2\theta)d\theta$	A1
	$= \frac{1}{2}\pi a^2 + a^2\int_0^{\frac{\pi}{2}}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$ (Use of double angle formula.)	M1
	$= \frac{1}{2}\pi a^2 + a^2\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$ (Integration)	M1
	$= \frac{1}{2}\pi a^2 + a^2\left(\frac{3}{4}\pi - 2\right) = \left(\frac{5}{4}\pi - 2\right)a^2$ (AG)	A1 (6) **[11]**

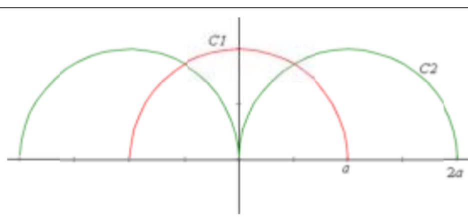
Q5.

5	$\sin\frac{1}{2}\theta = \frac{1}{2} \Rightarrow \theta = \frac{1}{3}\pi$; Intersection at $\left(\frac{1}{\sqrt{2}}, \frac{1}{3}\pi\right)$ (Accept 1.05 for $\frac{1}{3}\pi$.)	M1A1 (2)
	C_1: Circle centre at pole and radius $1/\sqrt{2}$ C_2: Curve, approx. correct orientation, from (0,0) to $(1,\pi)$. Completely correct correct shape.	B1 B1 B1 (3)
	$\frac{1}{6}\times\pi\times\frac{1}{2} - \frac{1}{2}\int_0^{\frac{1}{3}\pi}\sin\frac{1}{2}\theta\,\mathrm{d}\theta$	M1A1
	$= \frac{1}{12}\pi\times\frac{1}{2}\left[-2\cos\frac{1}{2}\theta\right]_0^{\frac{1}{3}\pi} = \frac{1}{12}\pi + \frac{\sqrt{3}}{2} - 1$	M1A1 (4) **Total: 9**

Q6.

	Using $x = r\cos\theta$ **and** $y = r\sin\theta$	**B1**
	$r^2 = 8\operatorname{cosec}2\theta \Rightarrow r^2 = \frac{4}{\sin\theta\cos\theta}$	**M1**
	$\Rightarrow r\cos\theta . r\sin\theta = 4 \Rightarrow xy = 4$ (in simple form)	**A1** [3]
	Sketch: Curve in 1st quadrant with correct concavity, asymptotic to **both** axes.	**B1B1** [2]
	$\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 8\operatorname{cosec}2\theta\,\mathrm{d}\theta = \left[2\ln\lvert\tan\theta\rvert\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$	**M1A1**
	$= 2\left\{\ln\lvert\sqrt{3}\rvert - \ln\left\lvert\frac{1}{\sqrt{3}}\right\rvert\right\}$ =2 ln3orln9	**A1** [3]

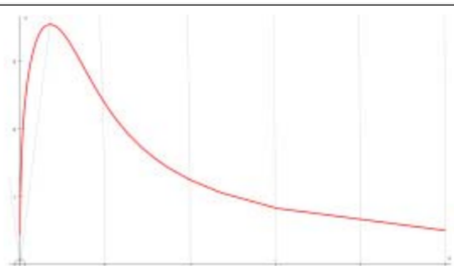
Q7.

(i)	$a = 2a\lvert\cos\theta\rvert \Rightarrow \cos\theta = \pm\frac{1}{2}$	**M1**	Eliminates r .
	$\left(a, \frac{\pi}{3}\right)$ and $\left(a, \frac{2\pi}{3}\right)$	**A1**	Both points needed for A1.
(ii)		**B1**	Semicircle for C1 including r = a.
		B1	Half of C2 including r = 2a.
		B1	Other half of C2 and line of symmetry.

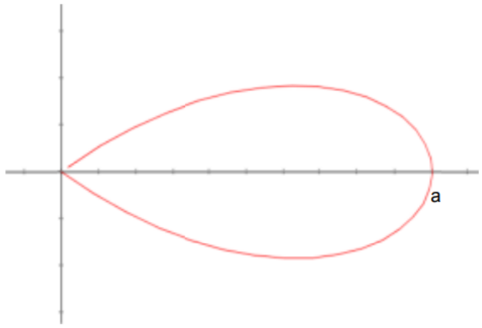
(iii)	$4a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$	**M1**	Finds area of segment OP_1 and OP_2 of C_2
	$=2a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos 2\theta+1\,d\theta$	**M1**	Uses $\cos^2\theta=\frac{1}{2}(\cos 2\theta+1)$
	$=2a^2\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}= 2a^2\left(\frac{\pi}{2}-\left(\frac{\sqrt{3}}{4}+\frac{\pi}{3}\right)\right)= a^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$	**A1**	Integrates correctly.
	Area $=\frac{\pi a^2}{6}-a^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)=-\frac{\pi a^2}{6}+\frac{a^2\sqrt{3}}{2}$	**M1 A1FT**	M1 for subtracting 'their' OP_1P_2 from $\frac{\pi a^2}{6}$
		10	

Q8.

(i)	$\frac{25}{2}\int_{0.01}^{\frac{\pi}{2}}\cot\theta\,d\theta$	**M1**	Uses $\frac{1}{2}\int r^2\,d\theta$.
	$=\frac{25}{2}[\ln\sin\theta]_{0.01}^{\frac{\pi}{2}}$	**A1**	
	$=-\frac{25}{2}\ln\sin 0.01\approx 57.6$	**A1**	
		3	
(ii)	$y=5\cos^{\frac{1}{2}}\theta\sin^{\frac{1}{2}}\theta=\frac{5}{\sqrt{2}}\sin^{\frac{1}{2}}2\theta$	**M1**	Uses $y=r\sin\theta$.
	$\theta=0.01\Rightarrow y\approx 0.5$	**A1**	
		2	
(iii)	$\frac{dy}{d\theta}=\frac{5}{\sqrt{2}}\sin^{-\frac{1}{2}}2\theta\cos 2\theta=0$ or $\max(\sin 2\theta)=1$	**M1 A1**	Sets $\frac{dy}{d\theta}=0$ or considers max (AEF).
	$\Rightarrow y=\frac{5\sqrt{2}}{2}\ (=3.54)$	**A1**	
		3	

(iv)		**B1**	Intersecting the initial line only when $x=0$ and $y=0$.
		B1	Correct shape.
		2	

Q9.

(i)		B1	Just one loop, correct shape at extremities
		B1	Correct position including (a, 0) labelled or in table.
		2	
(ii)	$\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2 cos^2 3\theta \, d\theta$	M1	For using correct formula
	$\frac{a^2}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 6\theta + 1) \, d\theta$	M1	Using double angle formula correctly
	$= \frac{a^2}{4}\left[\frac{1}{6}\sin 6\theta + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi a^2}{12}$	A1	
		3	
(iii)	$r = a\cos\theta\left(4\cos^2\theta - 3\right) \Rightarrow r = a\left(\frac{x}{r}\right)\left(4\left(\frac{x}{r}\right)^2 - 3\right)$	B1	Uses $x = r\cos\theta$ **and** $x^2 + y^2 = r^2$.
	$\Rightarrow r^4 = ax\left(4x^2 - 3r^2\right) \Rightarrow \left(x^2 + y^2\right)^2 = ax\left(4x^2 - 3\left(x^2 + y^2\right)\right)$	M1	For eliminating θ
	$\Rightarrow \left(x^2 + y^2\right)^2 = ax\left(x^2 - 3y^2\right)$	A1	Any equivalent cartesian form without fractions.
		3	