Matrices 2 MS

Q1.

7(a)	1, -1, 2	B1	
		1	
7(b)	$\mathbf{P}^3 - 2\mathbf{P}^2 - \mathbf{P} + 2\mathbf{I} = 0$	B1	States that **P** satisfies its characteristic equation.
	$2\mathbf{P}^{-1} = \mathbf{I} + 2\mathbf{P} - \mathbf{P}^2$	M1	Multiplies through by $\mathbf{P}^{-1}$.
	$\mathbf{P}^2 = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} 1 & 4 & -3 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	M1 A1	
		4	
7(c)	$\mathbf{A} = \mathbf{P}\begin{pmatrix} b & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{P}^{-1}$	M1	Applies $\mathbf{A} = \mathbf{PDP}^{-1}$
	$= \begin{pmatrix} b & -4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & -3 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$	M1 A1	Multiplies two adjacent matrices.
	$\begin{pmatrix} b & 4b+4 & -3b-1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	A1	
		4	

Q2.

9(a)	$\begin{vmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 4a \neq 0$ or $\begin{matrix} x = \frac{11}{2} + \frac{8}{a} \\ y = -\frac{1}{2} \\ z = -\frac{9}{2} \end{matrix}$	M1 A1	Shows that determinant is non-zero or solves equations and finds at least two correct values.
	The three planes intersect at a single point.	B1	
		3	
9(b)	$\begin{vmatrix} a-\lambda & 2a+5 & a+1 \\ 0 & -4-\lambda & 0 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0$	M1	Equates determinant to zero.
	$(a-\lambda)(-4-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda = a, -1, -4$	A1	AG
		2	

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9(c)	$\lambda = a: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4-a & 0 \\ 0 & 3 & -1-a \end{vmatrix} = \begin{pmatrix} (-4-a)(-1-a) \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	**M1 A1**	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda = -1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a+1 & 2a+5 & a+1 \\ 0 & -3 & 0 \end{vmatrix} = \begin{pmatrix} 3(a+1) \\ 0 \\ -3(a+1) \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	**A1**	
	$\lambda = -4: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a+4 & 2a+5 & a+1 \\ 0 & 3 & 3 \end{vmatrix} = \begin{pmatrix} 3a+12 \\ -(3a+12) \\ 3a+12 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	**A1**	
	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$	**A1**	OE
		5	

Q3.

1(a)	$\begin{vmatrix} a & 3 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & -5 \end{vmatrix} = -11a + 26$	**M1 A1**	Finds determinant. If solving equations x, y, z in terms in a.
	$\det \mathbf{A} \neq 0$ leads to unique solution.	**A1**	States that $\det \mathbf{A} \neq 0$. (Since a is an integer.) (M1 A0 A0 if one of x, y, z found in terms of a.) (M1 A1 A0 if two of x, y, z found in terms of a.)
	The three planes intersect at a single point.	**B1**	
		4	
1(b)	$a = 4$	**B1**	
		1	

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Q4.

6(a)	Eigenvalues of $\mathbf{A}$ are 5, -6 and 1.	B1	Upper diagonal matrix or characteristic equation.
	$\lambda=5:\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{22}{3} & 8 \\ 0 & -11 & 0 \end{vmatrix}=\begin{pmatrix}88\\0\\0\end{pmatrix}\sim\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\lambda=-6:\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 11 & -\frac{22}{3} & 8 \\ 0 & 0 & 7 \end{vmatrix}=\begin{pmatrix}-\frac{154}{3}\\-77\\0\end{pmatrix}\sim\begin{pmatrix}2\\3\\0\end{pmatrix}$ $\lambda=1:\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -\frac{22}{3} & 8 \\ 0 & -7 & 0 \end{vmatrix}=\begin{pmatrix}56\\0\\-28\end{pmatrix}\sim\begin{pmatrix}2\\0\\-1\end{pmatrix}$	M1 A1 A1 A1	Uses vector product (or equations) to find corresponding Eigenvectors. A1 for each correct Eigenvector.
	Thus $\mathbf{P}=\begin{pmatrix}1&2&2\\0&3&0\\0&0&-1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}25&0&0\\0&36&0\\0&0&1\end{pmatrix}$	M1 A1	Or correctly matched permutations of columns. (Accept scalar multiples of the eigenvectors shown here.) $\mathbf{P}$ must have at least two non-zero columns.
		7	
6(b)	$(\lambda-5)(\lambda+6)(\lambda-1)=\lambda^3-31\lambda+30=0\ \ =0$	B1	Finds characteristic equation.
	$\mathbf{A}^3-31\mathbf{A}+30\mathbf{I}=\mathbf{0}\Rightarrow\mathbf{A}^3=31\mathbf{A}-30\mathbf{I}$	M1	Finds $\mathbf{A}^3$ in terms of $\mathbf{A}$. Allow missing $\mathbf{I}$.
	$\mathbf{A}^3=\begin{pmatrix}125&-\frac{682}{3}&248\\0&-216&0\\0&0&1\end{pmatrix}$	M1A1	Substitutes for $\mathbf{A}$.
		4	

Q5.

2	$(\lambda+1)(\lambda-1)(\lambda-3)=\lambda^3-3\lambda^2-\lambda+3[=0]$	M1 A1	Finds characteristic equation.
	$\mathbf{A}^3-3\mathbf{A}^2-\mathbf{A}+3\mathbf{I}=0\Rightarrow\mathbf{A}^4-3\mathbf{A}^3-\mathbf{A}^2+3\mathbf{A}=0$	M1 A1	Substitutes $\mathbf{A}$ and multiplies through by $\mathbf{A}$.
	$\mathbf{A}^4=3(3\mathbf{A}^2+\mathbf{A}-3\mathbf{I})+\mathbf{A}^2-3\mathbf{A}=10\mathbf{A}^2-9\mathbf{I}$	M1 A1	Substitutes $\mathbf{A}^3$.
		6	

Q6.

8(a)	$\begin{vmatrix}13&18&-28\\-4&-a&8\\2&6&-5\end{vmatrix}=9a-24$	M1	Finds determinant.
	$a=\frac{8}{3}$	A1	Accept $\frac{24}{9}$.
		2	
8(b)	$\begin{pmatrix}13&18&-28\\-4&-1&8\\2&6&-5\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\-1\end{pmatrix}\Rightarrow\lambda=-1.$	B1	
		1	

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8(c)	$\begin{vmatrix} 13-\lambda & 18 & -28 \\ -4 & -1-\lambda & 8 \\ 2 & 6 & -5-\lambda \end{vmatrix} = 0$	B1	Sets $\det(\mathbf{A}-\lambda\mathbf{I})=0$.
	$-\lambda^3+7\lambda^2-7\lambda-15=0 \Rightarrow (\lambda+1)(\lambda-3)(\lambda-5)=0$	M1	Expands determinant and factorises.
	$\lambda=-1,\quad \lambda=3,\quad \lambda=5$	A1	
	$\lambda=3$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -4 & 8 \\ 2 & 6 & -8 \end{vmatrix} = \begin{pmatrix} -16 \\ -16 \\ -16 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	M1	Uses vector product (or equations) to find corresponding eigenvectors.
	$\lambda=5$: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -6 & 8 \\ 2 & 6 & -10 \end{vmatrix} = \begin{pmatrix} 12 \\ -24 \\ -12 \end{pmatrix} \sim \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$	A1 A1	A1 for each correct Eigenvector.
	Thus $\mathbf{P}=\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$	M1 A1	Or correctly matched permutations of columns. Accept scalar multiples of eigenvectors. At least two (non-zero) Eigenvectors for M1.
		8	

8(d)	$-\mathbf{A}^3+7\mathbf{A}^2-7\mathbf{A}-15\mathbf{I}=\mathbf{0}$	M1	Substitutes $\mathbf{A}$ into characteristic equation and multiplies through by $\mathbf{A}^{-1}$. Allow missing $\mathbf{I}$.
	$\mathbf{A}^{-1}=\frac{1}{15}\left(-\mathbf{A}^2+7\mathbf{A}-7\mathbf{I}\right)$	A1	Accept with $\mathbf{A}=\begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}$ and $\mathbf{A}^2=\begin{pmatrix} 41 & 48 & -80 \\ -32 & -23 & 64 \\ -8 & 0 & 17 \end{pmatrix}$ substituted to give $\mathbf{A}^{-1}=\frac{1}{15}\begin{pmatrix} 43 & 78 & -116 \\ 4 & 9 & -8 \\ 22 & 42 & -59 \end{pmatrix}$.
		2	

Q7.

2(a)	$\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & -3 \\ 1 & -1 & 7 \end{vmatrix} = -7-3+10-2(0)=0$	M1 A1	Shows that determinant is zero.
		2	
2(b)	$\begin{aligned} x-\ y+\ 2z &= 4, \\ x-\ y-\ 3z &= -5, \\ x-\ y+\ 7z &= 13, \end{aligned} \quad \Rightarrow z=\frac{9}{5},\ x-y=\frac{2}{5}$	M1 A1	M1 for row operations or eliminating one variable. A1 for deriving one correct equation with two unknowns.
	The three planes form a sheaf.	B1	All three planes have the same common line.
		3	

2(c)	$\begin{aligned} x-\ y+\ 2z &= 4, \\ x-\ y-\ 3z &= a, \\ x-\ y+\ 7z &= 13, \end{aligned} \quad \Rightarrow z=\frac{9}{5}, 3z+a=4-2z \Rightarrow a=-5$	B1	Derives contradiction.
	The three planes form a triangular prism.	B1	
		2	